Therefore a(n) = p(n - 1) + p(n - 2) + ... + p(1) + 1.

Express the number of distinct parts in terms of p(1), p(2), ... , p(n - 1)

Now consider b(n), the sum, over all partitions of n, of the number of distinct parts.
Another way to look at this sum is to count the number of partitions of n in which the number i appears, and sum those results for $1 \le i \le n$.

The number of partitions of n in which i appears is p(n - i), for $1 \le i \le n-1$.
This follows as $a_1 + ... + i + ... + a_k = n \Leftrightarrow a_1 + ... + a_n = i$.
The special case of i = n adds one further case.

Therefore b(n) = p(1) + p(2) + ... + p(n - 1) + 1.

Therefore a(n) = b(n); the number of 1s is equal to the number of distinct parts.

Further reading

1. Integer partition
2. PDF A Combinatorial Miscellany by Anders Björner and Richard P. Stanley

Source: Richard P. Stanley